a man lost his wedding ring, and set out to search for it with the help of a friend and a metal detector. They did not find the ring, but they did find some gold coins from the time of King Henry VIII, delivering 100 000 pounds to the two friends. The one-pound coins were preserved in very good condition, and during the 500 years elapsed, their average annual increase in value was between 1.42% and 1.43%. How many coins may they have found? **Exercises upwards of grade 11: C. 1593.** Two sides of a triangle are 3 cm and 4 cm long. What is the angle enclosed by the sides if the medians drawn from the opposite vertices are perpendicular to each other? **C. 1594.** The first row of seats in an auditorium consists of 24 seats. 20 seats are already taken. What is the probability that there are 2 adjacent vacant seats?

New exercises – competition B (see page 97): **B. 5078.** Define a sequence $a_1, a_2, \dots$ by the following recurrence relation: $a_1 = 1$, $a_n = \frac{n+1}{n-1}(a_1 + a_2 + \dots + a_{n-1})$ for $n > 1$. Determine the value of a_{2020}. (*4 points*) **B. 5079.** Solve the equation $\log_2 \log_3 x + \log_3 \log_2 x = \log_2 \frac{6}{\log_2 3}$ over the set of real numbers. (*3 points*) (Proposed by *B. Bíró,* Eger) **B. 5080.** Let D denote the midpoint of the base AB in an isosceles triangle ABC. Let H be the point lying closer to C that divides the leg AC in a $1:2$ ratio. The circle BCH intersects line CD at the points C and X. Show that $CX = \frac{4}{3}r$, where r is the radius of the circle ABC. (*4 points*) **B. 5081.** In a triangle, the medians drawn to sides a and b are perpendicular. Prove that $\frac{1}{2} < \frac{a}{b} < 2$. (*3 points*) **B. 5082.** Prove that the geometric mean, the arithmetic mean, and the quadratic mean of the altitudes in any triangle is not greater than the geometric mean, the arithmetic mean, and the quadratic mean of the radii of the escribed circles, respectively. (*5 points*) **B. 5083.** Is there a polynomial $p(x)$ of degree 100 with real coefficients for which the polynomial $p\big(p(x)\big)$ has 10000 distinct real roots? (*5 points*) **B. 5084.** Let n be a positive integer, and let $\mathcal{S}$ denote the set of $0-1-2$ sequences of length n. Determine which sets $\emptyset \neq A \subseteq \mathcal{S}$ have the following property: no matter how a vector $(c_1, c_2, \dots, c_n) \in \mathcal{S} \setminus \big\{(0, 0, \dots, 0)\big\}$ is selected, the probabilities that the sum of products $c_1a_1 + c_2a_2 + \dots + c_na_n$ formed with a randomly chosen element $(a_1, a_2, \dots, a_n)$ of set A will leave a remainder of 0, 1, or 2 are 1/3 each. (*6 points*) (Based on a problem of *Kürschák competition*) **B. 5085.** Show that a regular heptagon can be dissected into a finite number of symmetrical trapezoids, all similar to each other. (*6 points*) (Proposed by *M. Laczkovich,* Budapest)

New problems – competition A (see page 99): **A. 769.** Find all triples (a, b, c) of distinct positive integers so that there exists a subset S of the positive integers for which for all positive integers n exactly one element of the triple (an, bn, cn) is in S. (Proposed by *Carl Schildkraut,* Massachussets Institute of Technology) **A. 770.** Find all positive integers n such that $n!$ can be written as the product of two Fibonacci numbers. **A. 771.** Let ω denote the incircle of triangle ABC, which is tangent to side BC at point D. Let G denote the second intersection of line AD and circle ω. The tangent to ω at point G intersects sides AB and AC at points E and F. The circumscribed circle of DEF intersects ω at points D and M. The circumscribed circle of BCG intersects ω at point G and N. Prove that lines AD and MN are parallel. (Proposed by *Ágoston Győrffy,* Remeteszőlős)

Problems in Physics

(see page 121)

M. 393. Hang a piece of twisted rope or yarn, and then attach different weights to its free end. Leave the rope unwind, continuously slowing the motion with your hand until it reaches the equilibrium position. How does the angle turned by the lower end of the rope depend on the load?

G. 697. We look into a kaleidoscope; *part* of the observed view is shown in the *figure.* Where are the mirrors of the kaleidoscope? **G. 698.** We have three solid cubes of edges 1 cm, 3 cm and 9 cm. A tower is built from the three cubes. By what factor will the pressure exerted by the tower on the horizontal tabletop be greater when the smallest cube is at the bottom compared to when the largest cube is at the bottom? **G. 699.** At railway stations, we can often observe that the overhead contact wires are stretched with heavy weights suspended by wire ropes looped over, for example, a pulley system shown in the *figure.* *a*) Why is this method better than fixing the overhead contact lines? *b*) What is the tension in each of the double overhead contact wires if the total mass of the weights is 300 kg? *c*) How does the position of the weights change during a bright cloudless day from dawn to dusk? **G. 700.** The 315 billion-tonne D28 iceberg (also known as Loose tooth, or Molar Berg) broke off Antarctica on 25 September 2019. If it was smashed to small ice-cubes and put into water at a temperature of 20 °C, how many times of the amount of the water in lake Balaton could be cooled down to a temperature of 0 °C? The volume of the water in lake Balaton is 1.9 km^3. Suppose that the temperature of the ice-berg is -10 °C everywhere.

P. 5197. As a present, Winnie-the-Pooh was given a sphere-shaped balloon of radius 20 cm. The balloon was filled with helium such that when Winnie-the-Pooh released its thread, it just floated in the air, so it neither rose up nor sank. Winnie-the-Pooh was so happy that he began to scamper around in a circle with the balloon such that he held the end of the thread of the balloon in his hand. The balloon executed uniform circular motion. Piglet observed that whatever Winnie-the Pooh's constant angular speed was the thread of the balloon always had an angle of 45° with the tangent of the circular path of the balloon. What was the radius of the path of the balloon? (Neglect the weight of the thread and the incidental changes in the shape of the balloon.) **P. 5198.** There is a prism-shaped object of mass M on a horizontal smooth surface. One end of a light spring of spring constant D is attached to the prism such that the spring is parallel to the symmetry axis of the prism, whilst the other end is fixed to a disc-shaped bumper of negligible mass. Another object of mass m, sliding at a speed of v_0, collides with the bumper such that it partly compresses the long enough spring. *a*) What is the speed of the centre of mass of the system? *b*) Starting the stopwatch at the moment when the object touches the bumper first, how much time elapses until the spring becomes the shortest? **P. 5199.** A piece of thin and rigid metal wire of length ℓ has a shape of a circular arc. Each end of the wire is attached to a thin thread of length ℓ and the other ends of the threads are fixed at the centre of the circular arc, O, as shown in the *figure.* The period of the pendulum, which can swing with small amplitude in the vertical plane of the *figure,* is T_1. If the metal wire is straightened then the pendulum with the altered shape has a period of T_2, when it swings about O in the plane of the figure, with small amplitude. What is the ratio T_2/T_1? **P. 5200.** The speed of transverse waves in a suspension bridge over a valley is 400 m/s. In a storm the strong wind generated impulses, which are repeated in each second. What is the distance between the pillars if the bridge started to swing heavily? **P. 5201.** The *figure* shows a piston and two springs attached to it. The spring constant of both springs is $D = 1000$ N/m, the ambient air pressure is $p_0 = 10^5$ Pa, and the piston of cross-sectional area $A = 10$ dm^2 encloses a sample of monatomic gas. Initially both springs are unstretched, and the volume of the gas is $V_0 = 50$ litres. How much does the piston move, if $Q = 2$ kJ thermal energy is added to the sample of gas? (The walls of the container and the piston are thermally insulated; friction, and the heat capacity of the heating element are negligible.) **P. 5202.** The specific heat capacity of metals at very low temperatures is approximately proportional to the absolute temperature ($c = \alpha \cdot T$, where the proportionality constant α is characteristic of the material). In a very well insulated chamber of a cryogenic laboratory, two pieces of different metals of different mass are placed such that they came into contact. The initial temperature of one of them (denoted by A) is 1.0 K, whilst that of the other (B) is 3.0 K, and the final common temperature is 2.0 K. What will the final common temperature be if the initial temperature values of the metals are $T_A = 1.5$ K and $T_B = 2.5$ K? **P. 5203.** The refractive index of a transparent glass sheet of width $2A$ varies in the direction of axis z, which is perpendicular to the plane of the sheet. At $z = \pm A$ its value is n_0, whilst at $z = 0$ it is n_1. A thin ray of laser enters into the glass (at a "height of" $z = A$) and travels in the direction of axis x. The laser beam is deflected in the glass and travels along the curved path of a cosine function. *a*) How does the refractive index depend on z? *b*) What is the wavelength of the path of the laser? *Data:* $A = 1$ cm, $n_0 = 1.5$ and $n_1 = 1.6$. **P. 5204.** Determine the sensitivity of the cathode-ray oscilloscope shown in the *figure* in the unit of mm/volt. *Data:* the length of the deflecting plates is $\ell = 2$ cm, their distance is $d = 0.5$ cm, the distance between the screen and the centre of the plates is $s = 20$ cm, the accelerating voltage is $U_0 = 1000$ V, and the greatest value of the deflecting voltage is $U_{\max} = 100$ V. **P. 5205.** The *figure* shows a loop made of a piece of copper wire. The shape of the loop is two concentric semi-circles and two connecting straight line segments. The loop is on a horizontal tabletop, but initially the smaller semi-circle is in a vertical position. The small semi-circle is turned into the horizontal position in 1 s. The dashed line is the axis of rotation. The whole loop is in uniform vertically upward magnetic field. *a*) In which case is the flux linkage of the loop greater? *b*) What is the average value, and the direction of the induced current in the loop, while the smaller loop turns? What is the direction of the current? *c*) What is the greatest value of the induced current if the small semicircle is rotated at a constant angular speed and it takes exactly $\Delta t = 1$ s to turn from the vertical position to the horizontal position? *Data:* the magnetic induction is $B = 0.35$ T, the resistance of the loop is $R = 0.025\ \Omega$, the radius of the smaller semi-circle is $r = 0.2$ m. **P. 5206.** Determine the atomic mass number of ionium, which is the daughter element of uranium, after the uranium emits two α and two β particles. Which is the element whose isotope is the ionium? **P. 5207.** Muon (μ^-) is an unstable elementary particle, its mean lifetime is 2.197 μs, its mass is 207 times the mass of an electron, and its charge is the same as the charge of an electron. In a storage ring (a type of circular particle accelerator) there is uniform magnetic field, which is perpendicular to the plane of the ring. At a certain point of the ring, from the direction of the tangent at that point, a mono-energetic muon beam is injected into the storage ring. The muons revolve along the circular path and on average they decay after completing five whole turns. *a*) What is the (average) speed and kinetic energy of the muons if the radius of the storage ring is 120 m? *b*) What is the magnetic induction in the storage ring?

Problems of the 2019 Kürschák competition

1. In the acute triangle ABC we have $AB < AC < BC$. Let A_1, B_1 and C_1 be the feet of the altitudes from A, B and C, respectively. The point P is obtained by reflecting C_1 over the line BB_1 and the point Q is obtained by reflecting B_1 over the line CC_1. Prove that the circumcircle of the triangle A_1PQ passes through the midpoint of the side BC.

2. Let n be a positive integer. Find all families $\mathcal{F}$ that consist of certain subsets of $\{1, 2, \ldots, n\}$ and satisfy that for every fixed, nonempty subset $X \subseteq \{1, 2, \ldots, n\}$, the number of sets $A \in \mathcal{F}$ yielding an intersection $A \cap X$ of even, resp. odd cardinality is the same.

3. Is it true that for any bounded subsets H and A of the real line, the set H can be partitioned into pairwise disjoint translates of A in at most one way? (Infinitely many translates may be used.)